

THE BRAUER GROUP OF AN AFFINE DOUBLE PLANE ASSOCIATED TO A HYPERELLIPTIC CURVE

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ABSTRACT. For an affine double plane defined by an equation of the form $z^2 = f$ we study the divisor class group and the Brauer group. Two cases are considered. In the first case, f is a product of n linear forms in $k[x, y]$ and X is birational to a ruled surface $\mathbb{P}^1 \times C$, where C is rational if n is odd and hyperelliptic if n is even. In the second case, f is the equation of an affine hyperelliptic curve. On the open set where the cover is unramified, we compute the groups of divisor classes, the Brauer groups, the relative Brauer group, as well as all of the terms in the exact sequence of Chase, Harrison and Rosenberg.

1. INTRODUCTION

1.1. Statement of main results. Throughout k is an algebraically closed field of characteristic different from 2. This article is concerned with the study of divisor classes and algebra classes on a double cover X of the affine plane \mathbb{A}^2 . The surface X which we investigate is a hypersurface in \mathbb{A}^3 which is defined by an equation of the form $z^2 = f$.

In Section 2, we investigate the surface X which is defined by an equation of the form $z^2 = f_1 \cdots f_n$, where each f_i is a linear form in $k[x, y]$. The surface X is birational to a ruled surface $\mathbb{P}^1 \times C$, where C is rational if n is odd and hyperelliptic if n is even. There is an isolated singular point at the origin.

In Section 3, the surface X is defined by an equation of the form $z^2 = f$, where $f = y^2 - p(x)$. The zero set of f in \mathbb{A}^2 is denoted F . If $p(x)$ has degree at least three and is square-free, then F is an affine hyperelliptic curve. The surface X is rational, normal, and any singularity on X is a rational double point of type A_n .

The invariants of the surface X which we compute include the group of Weil divisor classes $\text{Cl}(\cdot)$, the Picard group $\text{Pic}(\cdot)$, and the Brauer group $\text{B}(\cdot)$. Let $K \rightarrow L$ be the quadratic Galois extension with group G of the rational function fields of \mathbb{A}^2 and X . On an open affine subset, the double cover $X \rightarrow \mathbb{A}^2$ is unramified, let $R \rightarrow S$ be the corresponding quadratic Galois extension of commutative rings. We are especially interested in the relative Brauer group $\text{B}(S/R)$ and the image of $\text{B}(R) \rightarrow \text{B}(L)$. For the Galois extension $R \rightarrow S$, the terms in the Chase, Harrison, Rosenberg cohomology sequence [5, Corollary 5.5] are computed. It is shown that the relative Brauer group $\text{B}(S/R)$ is isomorphic to $H^1(G, \text{Cl}(X))$. In [11] this relationship is investigated further, where it is demonstrated that every element of $\text{B}(S/R)$ is represented by a generalized crossed product algebra of degree two over $k[x, y]$.

To simplify the exposition, in Section 4 we include some computations involving divisors on a hyperelliptic curve.

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Preliminary Report.

1.2. Background material. We recommend [22] as a source for all unexplained notation and terminology. We tacitly assume all groups and sequences of groups are ‘modulo the characteristic of k ’. By μ_d we denote the kernel of the d th power map $k^* \rightarrow k^*$. By μ we denote $\cup_d \mu_d$. There is an isomorphism $\mathbb{Q}/\mathbb{Z} \cong \mu$, which is non-canonical and when convenient we use the two groups interchangeably. Cohomology and sheaves are for the étale topology, except where we use group cohomology. For instance, \mathcal{O} denotes the sheaf of rings and \mathbb{G}_m denotes the sheaf of units. If X is a variety over k , then the multiplicative group of units on X is $H^0(X, \mathbb{G}_m)$ which is also denoted $\mathcal{O}^*(X)$, or simply X^* . The Picard group [22, Proposition III.4.9], $\text{Pic}(X)$, is given by $H^1(X, \mathbb{G}_m)$. The torsion subgroup of $H^2(X, \mathbb{G}_m)$ is the cohomological Brauer group [14], denoted $B'(X)$. The Brauer group $B(X)$ of classes of $\mathcal{O}(X)$ -Azumaya algebras embeds into $B'(X)$ in a natural way [14, (2.1), p. 51] and for all varieties considered in this article $B(X) = B'(X)$ [18]. By Kummer theory, the d th power map

$$(1) \quad 1 \rightarrow \mu_d \rightarrow \mathbb{G}_m \xrightarrow{d} \mathbb{G}_m \rightarrow 1$$

is an exact sequence of sheaves on X . For any abelian group M and integer d , by ${}_d M$ we denote the subgroup of M annihilated by d . The long exact sequence of cohomology associated to (1) breaks up into short exact sequences which in degrees one and two are

$$(2) \quad 1 \rightarrow X^*/X^{*d} \rightarrow H^1(X, \mu_d) \rightarrow {}_d \text{Pic} X \rightarrow 0$$

$$(3) \quad 0 \rightarrow \text{Pic} X \otimes \mathbb{Z}/d \rightarrow H^2(X, \mu_d) \rightarrow {}_d B(X) \rightarrow 0$$

The group $H^1(X, \mu_d)$ classifies the Galois coverings $Y \rightarrow X$ with cyclic Galois group \mathbb{Z}/d [22, pp. 125–127]. We will utilize the following form of the exact sequence of Chase, Harrison and Rosenberg [5, Corollary 5.5]. If $Y \rightarrow X$ is a Galois covering with finite cyclic group G , and $\text{Pic} X = 0$, then there is an exact sequence of abelian groups.

$$(4) \quad 0 \rightarrow (\text{Pic} Y)^G \rightarrow H^2(G, Y^*) \xrightarrow{\alpha_4} B(Y/X) \xrightarrow{\alpha_5} H^1(G, \text{Pic} Y) \rightarrow 0.$$

If X is a nonsingular integral affine surface over k with field of rational functions $K = K(X)$, then there is an exact sequence

$$(5) \quad 0 \rightarrow B(X) \rightarrow B(K) \xrightarrow{a} \bigoplus_{C \in X_1} H^1(K(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{p \in X_2} \mu(-1) \xrightarrow{s} 0$$

The first summation is over all irreducible curves C on X , the second summation is over all closed points p on X . Sequence (5) is obtained by combining sequences (3.1) and (3.2) of [2, p. 86]. The map a of (5) is called the *ramification map*. Let Λ be a central K -division algebra which represents a class in $B(K)$. The curves $C \in X_1$ for which $a([\Lambda])$ is non-zero make up the so-called *ramification divisor* of Λ on X . For α, β in K^* , by $\Lambda = (\alpha, \beta)_d$ we denote the symbol algebra over K of degree d . Recall that Λ is the associative K -algebra generated by two elements, u and v subject to the relations $u^d = \alpha$, $v^d = \beta$, $uv = \zeta vu$, where ζ is a fixed primitive d th root of unity. On the Brauer class containing $(\alpha, \beta)_d$, the ramification map a agrees with the so-called *tame symbol*. Let C be a prime divisor on X . Then $\mathcal{O}_{X,C}$ is a discrete valuation ring with valuation denoted by v_C . The residue field is $K(C)$, the field of rational functions on C . The ramification of $(\alpha, \beta)_d$ along C is the cyclic Galois extension of $K(C)$ defined by adjoining the d th root of

$$(6) \quad \alpha^{v_C(\beta)} \beta^{-v_C(\alpha)}.$$

If $(\alpha, \beta)_d$ has non-trivial ramification along C , then C is a prime divisor of (α) or (β) .

In the usual way, we consider the affine plane \mathbb{A}^2 as an open subset of the projective plane \mathbb{P}^2 . Let F_0, F_1, \dots, F_n be distinct curves in \mathbb{P}^2 each of which is simply connected,

that is, $H^1(F_i, \mathbb{Q}/\mathbb{Z}) = 0$. Let $Z = \{F_i \cap F_j \mid i \neq j\}$, which is a subset of the singular locus of $F = F_0 + F_1 + \cdots + F_n$. Because each F_i is simply connected, if there is a singularity of F which is not in Z , then at that point F is geometrically unbranched. Decompose Z into irreducible components $Z_1 + \cdots + Z_s$. The graph of F is denoted Γ and is bipartite with edges $\{F_0, \dots, F_n\} \cup \{Z_1, \dots, Z_s\}$. An edge connects F_i to Z_j if and only if $Z_j \subseteq F_i$. So Γ is a connected graph with $n + 1 + s$ vertices. Let e denote the number of edges. Then $H_1(\Gamma, \mathbb{Z}/d) \cong (\mathbb{Z}/d)^{(r)}$, where $r = e - (n + 1 + s) + 1$. We will utilize the following special version of [9, Theorem 4].

Theorem 1.1. *Let f_1, \dots, f_n be irreducible polynomials in $k[x_1, x_2]$ defining n distinct curves F_1, \dots, F_n in the projective plane $\mathbb{P}^2 = \text{Proj } k[x_0, x_1, x_2]$. Let $F_0 = Z(x_0)$ be the line at infinity and Γ the graph of $F = F_0 + F_1 + \cdots + F_n$. If $H^1(F_i, \mathbb{Q}/\mathbb{Z}) = 0$ for each i , and $R = k[x_1, x_2][f_1^{-1}, \dots, f_n^{-1}]$, then for each $d \geq 2$, ${}_d B(R) \cong H_1(\Gamma, \mathbb{Z}/d)$. If F_i and F_j intersect at P_{ij} with local intersection multiplicity μ_{ij} , then near the vertex P_{ij} , the cycle in Γ corresponding to $(f_i, f_j)_d$ looks like $F_i \xrightarrow{\mu_{ij}} P_{ij} \xrightarrow{-\mu_{ij}} F_j$.*

Proof. See [9, Theorem 5] and [10, § 2]. If moreover we assume each F_i is a line, the proof of [9, Theorem 4] shows that the Brauer group ${}_v B(R)$ is generated by the set of symbol algebras $\{(f_i, f_j)_v \mid 1 \leq i < j < n\}$ over R and the only relations that arise are when three of the lines F_i meet at a common point of \mathbb{P}^2 . \square

2. A RULED SURFACE WITH AN ISOLATED SINGULARITY

Let $A = k[x, y]$ and $K = k(x, y)$ the quotient field of A . Let $f = f_1 f_2 f_3 \cdots f_n \in k[x, y]$ where $f_1, f_2, f_3, \dots, f_n$ are linear polynomials, all of the form $f_i = a_i x + b_i y$. Assume $n > 2$ and f is square-free in A . Set $T = A[z]/(z^2 - f)$, $R = A[f^{-1}]$ and $S = T[f^{-1}]$. It is an exercise [16, Exercise II.6.4] to show that T is an integral domain, a free A -module of rank 2, and is integrally closed in its quotient field, $L = K(z)$. The surface $X = \text{Spec } T$ has an isolated singularity at the maximal ideal (x, y, z) . The birational equivalence classes of such double planes have been classified in [4].

Because A is factorial, it is routine to see that $R^* = k^* \times \langle f_1 \rangle \cdots \times \langle f_n \rangle$. Let σ denote both the R -automorphism of S and the K -automorphism of L defined by $z \mapsto -z$. Let $G = \{1, \sigma\}$ be the group generated by σ which is a group of automorphisms of T , S and L . Notice that $A = T^G$, $R = S^G$, and $K = L^G$. Also, S/R and L/K are Galois by [22, Example I.3.4]. Because G is cyclic, the image of the homomorphism α_4 in (4) is generated by the symbol algebras $(f_i, f)_2$ for $i = 1, \dots, n$. We denote the image of α_4 by $B^\sim(S/R)$. By [9, Theorem 4], ${}_2 B(R) \cong (\mathbb{Z}/2)^{(n-1)}$ is generated by the symbol algebras $(f_i, f_j)_2$. Because all of the polynomials f_i are in the maximal ideal (x, y) , for any triple $i < j < l$, we have the Brauer equivalence $(f_i f_j, f_l)_2 \sim (f_i, f_j)_2$.

Proposition 2.1. *As above, let $T = k[x, y, z]/(z^2 - f)$, where $f_1, f_2, f_3, \dots, f_n$ are linear forms in $k[x, y]$, $n \geq 3$, and $f = f_1 f_2 f_3 \cdots f_n$ is square-free. Then T satisfies*

- (a) $T^* = k^*$,
- (b) $\text{Pic}(T) = 0$, and
- (c) $B(T) = 0$.

Proof. If n is even, we can define a grading on $k[x, y, z]$ by $\deg(x) = \deg(y) = 1$ and $\deg(z) = n/2$. Then $z^2 - f$ is a quasi-homogeneous (or weighted homogeneous) polynomial and T is a graded ring. The subring of T consisting of homogeneous elements of degree 0 is $T_0 = k$ from which we get (a). By [19], (b) and (c) follow. If n is odd, we

define a grading on $k[x, y, z]$ by $\deg(x) = \deg(y) = 2$ and $\deg(z) = n$. Then $z^2 - f$ is a homogeneous element of degree $2n$, and T is a graded ring with $T_0 = k$. \square

Proposition 2.2. *In the context of Proposition 2.1, the following are true.*

- (a) *If $n \geq 3$ is odd, then $B^\sim(S/R) = B(S/R) = {}_2B(R) \cong (\mathbb{Z}/2)^{(n-1)}$.*
- (b) *If $n \geq 4$ is even, then $B^\sim(S/R)$ has order two and is generated by the Brauer class of the product of symbols $(f_1, f_2)_2 \cdots (f_i, f_{i+1})_2 \cdots (f_{n-1}, f_n)_2$.*

Proof. The group ${}_2B(R)$ is generated by the symbols $(f_i, f_j)_2$. The subgroup $B^\sim(S/R)$ is generated by all of the symbols $(f_i, f)_2$. If n is odd,

$$\begin{aligned}
 (f_1 f_2, f)_2 &\sim (f_1 f_2, f_3 f_4 \cdots f_n)_2 \\
 &\sim (f_1 f_2, f_3)_2 \cdots (f_1 f_2, f_n)_2 \\
 &\sim (f_1, f_2)_2 \cdots (f_1, f_2)_2 \\
 &\sim (f_1, f_2)_2
 \end{aligned}
 \tag{7}$$

from which it follows that $B^\sim(S/R)$ contains all of the symbols $(f_i, f_j)_2$. Part (a) follows. Now assume n is even. If i is odd

$$\begin{aligned}
 (f_i, f)_2 &\sim (f_i, f_1 f_2)_2 \cdots (f_i, f_i f_{i+1})_2 \cdots (f_i, f_{n-1} f_n)_2 \\
 &\sim (f_1, f_2)_2 \cdots (f_i, f_{i+1})_2 \cdots (f_{n-1}, f_n)_2
 \end{aligned}
 \tag{8}$$

which is Brauer equivalent to

$$\begin{aligned}
 (f_i, f)_2 &\sim (f_i, f_1 f_2)_2 \cdots (f_i, f_{i-1} f_i)_2 \cdots (f_i, f_{n-1} f_n)_2 \\
 &\sim (f_1, f_2)_2 \cdots (f_{i-1}, f_i)_2 \cdots (f_{n-1}, f_n)_2
 \end{aligned}
 \tag{9}$$

if i is even. This implies that $B^\sim(S/R)$ is a group of order 2. \square

2.1. Quasi-homogeneous singularities. As shown in Proposition 2.1, the polynomial $h(x, y, z) = z^2 - f(x, y)$ is quasi-homogeneous and the ring T is graded. The singularity of X is a quasi-homogeneous singularity, where $X = Z(h)$ is viewed as a hypersurface in \mathbb{A}^3 . For $t \in k^*$, $h(t^2 x, t^2 y, t^n z) = t^{2n} h(x, y, z)$. Therefore, X is invariant under the k^* -action $(x, y, z) \mapsto (t^2 x, t^2 y, t^n z)$ on \mathbb{A}^3 . Using [24], we compute the desingularization of X . There are two cases.

If n is even, the singularity of X can be resolved by one blowing-up followed by a normalization. The exceptional divisor E is irreducible, isomorphic to a curve of genus $(n-2)/2$, and $E \cdot E = -2$.

If n is odd, the desingularization is accomplished by a sequence of one blowing-up, one normalization, then n more blowings-up. The exceptional divisor consists of $n+1$ copies of \mathbb{P}^1 , call them E_0, E_1, \dots, E_n . The intersection matrix is defined by

$$\begin{aligned}
 E_0 \cdot E_0 &= -(n+1)/2 \\
 E_0 \cdot E_1 &= \cdots = E_0 \cdot E_n = 1 \\
 E_1 \cdot E_1 &= \cdots = E_n \cdot E_n = -2 \\
 E_i \cdot E_j &= 0, \quad \text{if } 0 < i, 0 < j, i \neq j.
 \end{aligned}
 \tag{10}$$

The determinant of $(E_i \cdot E_j)$ is equal to 2^{n-1} . If we assume $k = \mathbb{C}$, then using [23] or [17], we compute $H_1(M, \mathbb{Z})$, the abelianized topological fundamental group of the intersection M of X with a small sphere about the singular point P . It follows that π_1 is generated by $\gamma_0, \gamma_1, \dots, \gamma_n$, with relations $e = \gamma_1 \cdots \gamma_n \gamma_0^{(n+1)/2}$, $e = \gamma_0 \gamma_1^{-2}$, \dots , $e = \gamma_0 \gamma_n^{-2}$, and γ_0 is

central. It follows that $H_1(M, \mathbb{Z})$ is isomorphic to the cokernel of the map defined by $(E_i \cdot E_j)$, hence is isomorphic to $(\mathbb{Z}/2)^{(n-1)}$.

Notice that for $n = 3$, the singularity of X is rational of type D_4 (see [21, p. 258] for instance). In [28, (3.5)] it is shown that the singularity of $X = \text{Spec } T$ is rational if and only if $n = 3$. For the remainder of this section we distinguish between the odd n and even n cases.

2.2. The case when n is odd. As above, let $f = f_1 f_2 f_3 \cdots f_n \in k[x, y]$ where each f_i is a linear form. For this section, we assume n is an odd integer greater than or equal to 3. Let $R = k[x, y][f^{-1}]$, $S = R[z]/(z^2 = f)$, and $T = k[x, y, z]/(z^2 = f)$. An affine change of coordinates allows us to assume $f_1 = x$ and for all $i > 1$, $f_i = a_i x - y$.

Proposition 2.3. *In the above notation, the following are true.*

- (a) S is a rational surface.
- (b) S is factorial, or in other words, $\text{Pic } S = 0$.
- (c) $S^* = k^* \times \langle z \rangle \times \langle f_2 \rangle \times \cdots \times \langle f_n \rangle$
- (d)

$$H^i(G, S^*) \cong \begin{cases} R^* & \text{if } i = 0 \\ \frac{\langle f_2 \rangle}{\langle f_2^2 \rangle} \times \cdots \times \frac{\langle f_n \rangle}{\langle f_n^2 \rangle} \cong (\mathbb{Z}/2)^{(n-1)} & \text{if } i = 2, 4, 6, \dots \\ \langle 1 \rangle & \text{if } i = 1, 3, 5, \dots \end{cases}$$

Proof. Consider the two k -algebra isomorphisms

$$(11) \quad S = \frac{k[x, y, z][z^{-1}]}{z^2 = f(x, y)} \xrightarrow{\alpha} \frac{k[x, y, w][w^{-1}]}{w^2 = x f(1, y)} \xrightarrow{\beta} k[v, w][w^{-1}, f(1, v)^{-1}]$$

where α is defined by $x \mapsto x$, $y \mapsto xv$, $z \mapsto x^{(n-1)/2}w$, and β is defined by $x \mapsto w^2 f(1, v)^{-1}$, $v \mapsto v$, $w \mapsto w$. Let U denote the ring on the right hand side of (11). Since U is rational and factorial, this proves (a) and (b). The group of units in U is equal to $k^* \times \langle w \rangle \times \langle f_2(1, v) \rangle \times \cdots \times \langle f_n(1, v) \rangle$. Using this, one checks that the group of units in S is generated by k^* and the elements z, f_2, \dots, f_n . Apply [25, Theorem 10.35] to get part (d). \square

Proposition 2.4. *In the context of Section 2.2, the following are true.*

- (a) $\text{Cl}(T) \cong (\mathbb{Z}/2)^{(n-1)}$ is generated by the prime divisors $I_i = (z, f_i)$, where $i = 2, \dots, n$.
- (b)

$$H^i(G, \text{Cl}(T)) \cong \begin{cases} \text{Cl}(T) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^{(n-1)} & \text{if } i \text{ is odd} \\ \text{Cl}(T)^G = \text{Cl}(T) \cong (\mathbb{Z}/2)^{(n-1)} & \text{if } i \text{ is even} \end{cases}$$

Proof. In T , the minimal primes of z are $I_1 = (f_1, z), \dots, I_n = (f_n, z)$. A local parameter for T_{I_i} is z . The divisor of z on $\text{Spec } T$ is $\text{div}(z) = I_1 + \cdots + I_n$. The only minimal prime of f_i is I_i . The divisor of f_i is $2I_i$. Since $S = T[z^{-1}]$, by Nagata's Theorem [12, Theorem 7.1], $\text{Cl}(T)$ is generated by I_1, \dots, I_n , subject to the relations $I_1 + \cdots + I_n \sim 0$, $2I_1 \sim 0, \dots, 2I_n \sim 0$. Therefore, $\text{Cl}(T) \cong (\mathbb{Z}/2)^{(n-1)}$ is generated by any $n-1$ of I_1, \dots, I_n and this proves (a). We remark that (a) also follows from [27] or [26]. Since the group $G = \langle \sigma \rangle$ acts trivially on $\text{Cl}(T)$, we get (b). \square

Proposition 2.5. *In the context of Proposition 2.3, the sequence*

$$0 \rightarrow B(S/R) \rightarrow B(R) \rightarrow B(S) \rightarrow 0$$

is exact.

Proof. Let U be as in the proof of Proposition 2.3. By [9, Theorem 4], the Brauer group ${}_m\mathbf{B}(U)$ is isomorphic to $(\mathbb{Z}/m)^{(n-1)}$. It is generated by the symbol algebras $(w, f_i(1, v))_m$. The Brauer group ${}_m\mathbf{B}(R)$ is isomorphic to $(\mathbb{Z}/m)^{(n-1)}$ and is generated by the symbol algebras $(x, f_i(x, y))_m$. Under the isomorphism $\beta\alpha$ in (11), the algebra $(x, f_i(x, y))_m$ maps to

$$(w^2 f_i(1, v)^{-1}, f_i(1, v))_m \sim (w^2, f_i(1, v))_m \sim (w, f_i(1, v))_m^2.$$

As a homomorphism of abstract groups, $\mathbf{B}(R) \rightarrow \mathbf{B}(S)$ can be viewed as “multiplication by 2”. \square

Remark 2.6. It follows from Proposition 2.3(a) that in (4), the sequence of Chase, Harrison, Rosenberg, the groups involving $\text{Pic} S$ are trivial. The map α_4 is an isomorphism.

Remark 2.7. Propositions 2.2 and 2.4 imply that $\mathbf{B}(S/R) \cong H^1(G, \text{Cl}(T))$.

In Proposition 2.8, let $X = \text{Spec } T$ and P the singular point of X . By a desingularization of P we mean a proper, surjective, birational morphism $\pi : \tilde{X} \rightarrow X$ such that \tilde{X} is nonsingular and π is an isomorphism on $\tilde{X} - E$, where $E = \pi^{-1}(P)$. Let $T^h = \mathcal{O}_P^h$ be the henselization of \mathcal{O}_P and $\hat{T} = \hat{\mathcal{O}}_P$ the completion.

Proposition 2.8. *The following are true for any desingularization $\pi : \tilde{X} \rightarrow X$.*

- (a) $\text{Cl}(X) = \text{Cl}(\mathcal{O}_P)$
- (b) $\text{Cl}(\mathcal{O}_P^h) = \text{Cl}(\hat{\mathcal{O}}_P)$ is isomorphic to $\text{Cl}(X) \oplus V$, where V is a finite dimensional k -vector space.
- (c) If $n = 3$, the singularity P is rational and $0 = \mathbf{B}(X) = H^2(X, \mathbb{G}_m)$.
- (d) If $n > 3$, the singularity P is irrational and $H^2(X, \mathbb{G}_m)$ is isomorphic to the k -vector space V of Part (b).
- (e) $0 = \mathbf{B}(\tilde{X}) = H^2(\tilde{X}, \mathbb{G}_m)$.
- (f) $0 = \mathbf{B}(X - P)$.

Proof. By [12, Corollary 10.3], $\text{Cl}(T) = \text{Cl}(\mathcal{O}_P)$. By Mori’s Theorem [12, Corollary 6.12], the natural maps $\text{Cl}(\mathcal{O}_P) \rightarrow \text{Cl}(\mathcal{O}_P^h) \rightarrow \text{Cl}(\hat{\mathcal{O}}_P)$ are one-to-one. By Artin Approximation [1], $\text{Cl}(\mathcal{O}_P^h) = \text{Cl}(\hat{\mathcal{O}}_P)$. Assume for now that $\pi : \tilde{X} \rightarrow X$ is a minimal resolution, so the exceptional divisor is as described in (10). The Neron-Severi group $\mathbf{E} = \bigoplus_{i=0}^n \mathbb{Z}E_i$ is the free abelian group on the irreducible components of the exceptional fiber. The cokernel of the intersection matrix (10) applied to \mathbf{E} is called $H(T)$. Then $H(T)$ is isomorphic to $(\mathbb{Z}/2)^{(n-1)}$. Given a divisor D on \tilde{X} , sending the class of D in $\text{Pic}(\tilde{X})$ to $\sum (D \cdot E_i)E_i$ defines a homomorphism $\theta : \text{Pic}(\tilde{X}) \rightarrow \mathbf{E}$. The kernel of θ is denoted $\text{Pic}^0(T)$. The cokernel of θ is denoted $G(T)$. By [21, Proposition 15.3], the groups $H(T)$, $\text{Pic}^0(T)$, $G(T)$ depend only on T and not \tilde{X} . By [21, Proposition 16.3], there is a commutative diagram

$$(12) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \text{Pic}^0(T) & \longrightarrow & \text{Cl}(T) & \longrightarrow & H(T) & \longrightarrow & G(T) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & \text{Pic}^0(\hat{T}) & \longrightarrow & \text{Cl}(\hat{T}) & \longrightarrow & H(\hat{T}) & \longrightarrow & 0 \end{array}$$

with exact rows. Let F_i denote the divisor on X corresponding to the prime ideal $I_i = (z, f_i)$. If \tilde{F}_i is the strict transform of F_i on \tilde{X} , then $\tilde{F}_i \cdot E_j = \delta_{ij}$, the Kronecker delta function. This shows that $G(T) = 0$. By Proposition 2.4(a), $\text{Cl}(T) \cong H(T)$ and $\text{Pic}^0(T) = 0$. From (12), $\text{Pic}^0(\hat{T}) \cong \text{Cl}(\hat{T}) / \text{Cl}(T)$. It follows from Artin’s construction in [3, pp. 486–488] that the group $\text{Pic}^0(\hat{T})$ is a finite dimensional k -vector space. For an exposition, see [13, pp. 423–426], especially the proof of Corollary 4.5 and the Remark which follows. This proves (b). The rest follows from [7]. \square

Question 2.9. As in Proposition 2.3, let $f = f_1 f_2 f_3 \cdots f_n \in k[x, y]$ where $f_1, f_2, f_3, \dots, f_n$ are linear forms, and n is an odd integer greater than 3. Assume $f_i \neq x$ and $f_i \neq y$ for all i . Let $K = k(x, y)$, $L = K(\sqrt{f})$. Consider $L_1 = L(\sqrt{x})$, $L_2 = L(\sqrt{y})$, $L_3 = L(\sqrt{x}, \sqrt{y})$. Then L_3/L is an abelian Galois extension of degree 2^2 . We also view L_3 as the function field of the surface S_3 which is defined by $z^2 = f(x^2, y^2)$. Can we use the extension S_3/S to reduce to the case where n is even?

2.3. The case when n is even. As above, let $T = k[x, y, z]/(z^2 - f)$, where $f_1, f_2, f_3, \dots, f_n$ are linear forms in $k[x, y]$, $f = f_1 f_2 f_3 \cdots f_n$ is square-free, and $n \geq 4$ is even. Without loss of generality assume $f_1 = x$ and $f_2 = y$. Write κ for $n/2$. As mentioned above, the singularity of T is resolved by blowing up the maximal ideal (x, y, z) and then taking the integral closure. The homomorphism of k -algebras

$$(13) \quad T = \frac{k[x, y, z]}{(z^2 - xyf_3f_4 \cdots f_n)} \rightarrow \frac{k[x, v, w]}{(w^2 - v(a_3 - b_3v)(a_4 - b_4v) \cdots (a_n - b_nv))} = \tilde{T}$$

defined by $x \mapsto x$, $y \mapsto xv$, $z \mapsto x^\kappa w$ is an affine piece of the resolution of T . Let C be the affine hyperelliptic curve defined by $w^2 - f(1, v)$. It is clear that $\tilde{T} = k[x] \otimes_k \mathcal{O}(C)$, hence is regular. Both T and \tilde{T} are integral domains. The map (13) is birational, because we identify $v = yx^{-1}$ and $w = zx^{-\kappa}$ with elements of the quotient field of T . So (13) is one-to-one.

Proposition 2.10. *If n is even, and S is as in the top of Section 2, then S is birational to a ruled surface $\mathbb{A}^1 \times C$, where C is an affine hyperelliptic curve defined by an equation of the form $y^2 = (x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_n)$. The genus of the nonsingular completion of C is $(n-2)/2$.*

Proof. The previous paragraph proves that T is birational to a ruled surface as claimed. But $S = T[z^{-1}]$ is birational to T . By Section 4.2, if \bar{C} is the complete nonsingular model for C , then the genus of \bar{C} is $\kappa - 1 = (n-2)/2$. \square

The ideal of \tilde{T} lying over (x, y, z) is the principal ideal (x) . Upon tensoring with $T[x^{-1}]$, (13) becomes an isomorphism. The only minimal prime of x in T is $I = (x, z)$. Upon localizing at I , z is a local parameter for T_I . The divisor of x on $\text{Spec } T$ is $2I$. By [12, Theorem 8.1], $\text{Cl}(\tilde{T}) = \text{Cl}(C)$. Because x is irreducible in \tilde{T} , it follows that $\text{Cl}(T[x^{-1}]) = \text{Cl}(\tilde{T}[x^{-1}]) = \text{Cl}(\tilde{T}) = \text{Cl}(C)$. The group $\mathcal{O}^*(C)$ of units on C is equal to k^* , because for a nonsingular completion \bar{C} of C there is only one point at infinity $\bar{C} - C$. It follows that the group of units of $T[x^{-1}]$ is $k^* \times \langle x \rangle$. Nagata's Theorem [12, Theorem 7.1] becomes

$$(14) \quad 0 \longrightarrow \mathbb{Z}I/2\mathbb{Z}I \longrightarrow \text{Cl}(T) \xrightarrow{\sigma} \text{Cl}(T[x^{-1}]) \longrightarrow 0.$$

Let T^h denote the henselization of T at the maximal ideal (x, y, z) . Let $X = \text{Spec } T$ and P the singular point of X . Let $T^h = \mathcal{O}_P^h$ be the henselization of \mathcal{O}_P and $\hat{T} = \hat{\mathcal{O}}_P$ the completion.

Proposition 2.11. *If $\pi : \tilde{X} \rightarrow X$ is any desingularization of P and $E = \pi^{-1}(P)$, then the following are true.*

- (a) $\text{Cl}(X) = \text{Cl}(\mathcal{O}_P)$.
- (b) $\text{Cl}(T^h) = \text{Cl}(\hat{T})$ is isomorphic to $\text{Cl}(X) \oplus V$, where V is a finite dimensional k -vector space.
- (c) $H^2(X, \mathbb{G}_m)$ is isomorphic to the k -vector space V of Part (b).
- (d) $0 = B(\hat{T}) = H^2(\hat{T}, \mathbb{G}_m)$.
- (e) $0 = B(\tilde{X}) = H^2(\tilde{X}, \mathbb{G}_m)$.

(f) $B(X - P)$ is isomorphic to $H^1(E, \mathbb{Q}/\mathbb{Z})$, which is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(n-2)}$.

Proof. Parts (a) and (b) are proved as in Proposition 2.8. By [15, Corollaire (1.2)], $B(C) = 0$. Since \tilde{T} is obtained from $\mathcal{O}(C)$ by adjoining an indeterminate, $B(\tilde{T}) = 0$. Since \tilde{T} is regular, $B(\tilde{T}) = H^2(\tilde{T}, G_m)$ by [18]. This proves (d). The rest follows from [7]. \square

For the Galois extension $R \rightarrow S$, Proposition 2.12 computes the terms in (4), the sequence of Chase, Harrison and Rosenberg.

Proposition 2.12. *Let f , T and S be as above. The following are true.*

(a) $B^\sim(S/R) = B(S/R)$ has order two.

(b) $\text{Pic}(S)$ is divisible and

$$H^i(G, \text{Pic}(S)) \cong \begin{cases} \text{Pic}(S) \otimes \mathbb{Z}/2 = 0 & \text{if } i \text{ is odd} \\ \text{Pic}(S)^G = {}_2\text{Pic}(S) \cong (\mathbb{Z}/2)^{(n-2)} & \text{if } i \text{ is even} \end{cases}$$

(c) $S^* = k^* \times \langle z \rangle \times \langle f_2 \rangle \times \cdots \times \langle f_n \rangle$

(d)

$$H^i(G, S^*) \cong \begin{cases} R^* & \text{if } i = 0 \\ \frac{\langle f_2 \rangle}{\langle f_2^2 \rangle} \times \cdots \times \frac{\langle f_n \rangle}{\langle f_n^2 \rangle} \cong (\mathbb{Z}/2)^{(n-1)} & \text{if } i = 2, 4, 6, \dots \\ \langle 1 \rangle & \text{if } i = 1, 3, 5, \dots \end{cases}$$

Proof. Let \tilde{T} denote the ring on the right hand side of (13), and $U = \tilde{T}[x^{-1}]$ the localized ring. Then $U = k[x, x^{-1}] \otimes \mathcal{O}(C)$, where C is the affine hyperelliptic curve $w^2 = v(a_3 - b_3v)(a_4 - b_4v) \cdots (a_n - b_nv)$. We know that $\text{Cl}(C) = \text{Cl}(\mathbb{A}^1 \times C)$. It follows that $\text{Cl}(U) = \text{Cl}(C)$. Proposition 4.1 says that

$$(15) \quad H^i(G, \text{Cl}(U)) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ {}_2\text{Cl}(C) \cong (\mathbb{Z}/2)^{(n-2)} & \text{if } i \text{ is even.} \end{cases}$$

By Nagata's Theorem [12, Theorem 7.1], $\beta : \text{Cl}(U) \rightarrow \text{Cl}(U[w^{-1}])$ is onto and the kernel of β is generated by those prime divisors containing w . In U there are $n-1$ minimal primes of w . They are $\{I_i = (w, f_i(1, v))\}_{i=2}^n$. The only minimal prime of $f_i(1, v)$ is I_i . Upon localizing U at I_i we see that w is a local parameter and the valuation of $f_i(1, v)$ is 2. The divisors of w and $f_i(1, v)$ are $\text{div}(w) = I_2 + \cdots + I_n$, $\text{div}(f_i(1, v)) = 2I_i$. The sequence

$$(16) \quad 1 \rightarrow U^* \rightarrow U[w^{-1}]^* \xrightarrow{\text{div}} \bigoplus_{i=2}^n \mathbb{Z}I_i \xrightarrow{\theta} \text{Cl}(U) \xrightarrow{\beta} \text{Cl}(U[w^{-1}]) \rightarrow 0$$

is exact. By Section 4.2, the kernel of β is the subgroup ${}_2\text{Cl}(U)$, an elementary 2-group of rank $n-2$. From this we see that the sequence

$$(17) \quad 1 \rightarrow U^* \rightarrow U[w^{-1}]^* \rightarrow \langle w \rangle \times \langle f_3(1, v) \rangle \times \cdots \times \langle f_n(1, v) \rangle$$

is split-exact. The ring S is isomorphic to the localization $U[w^{-1}]$, so part (c) follows from (17). Apply [25, Theorem 10.35] to get part (d). The sequence

$$(18) \quad 0 \rightarrow (\mathbb{Z}/2)^{(n-2)} \rightarrow \text{Cl}(U) \xrightarrow{\beta} \text{Cl}(U[w^{-1}]) \rightarrow 0$$

is exact. Since $\text{Cl}(C)$ is divisible, $\text{Cl}(U)$ is divisible, so β can be viewed as "multiplication by 2". On ${}_2\text{Cl}(U)$ the group G acts trivially. The long exact sequence of cohomology associated to (18) is

$$(19) \quad \cdots \xrightarrow{\partial^i} (\mathbb{Z}/2)^{(n-2)} \rightarrow H^i(G, \text{Cl}(U)) \xrightarrow{\beta^i} H^i(G, \text{Cl}(U[w^{-1}])) \xrightarrow{\partial^{i+1}} (\mathbb{Z}/2)^{(n-2)} \rightarrow \cdots$$

where β^i is the zero map. Part (b) follows from (15) and (19). Proposition 2.2 says $B^\sim(S/R)$ is a cyclic group of order two. Together with sequence (4), part (b), and part (d), this proves part (a). \square

Proposition 2.13. *In the context of Proposition 2.12, the following are true.*

(a) *There is an exact sequence*

$$0 \rightarrow \mathbb{Z}/2 \rightarrow {}_2B(R) \rightarrow {}_2B(T[f_1^{-1}]) \rightarrow 0$$

(b) *As an abstract group, $B(S)$ is isomorphic to $H^1(C, \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Q}/\mathbb{Z})^{(n-1)}$, which is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(n-2)} \oplus (\mathbb{Q}/\mathbb{Z})^{(n-1)}$.*

(c) *The cokernel of $B(R) \rightarrow B(S)$ is isomorphic to $H^1(C, \mathbb{Q}/\mathbb{Z})$.*

Proof. By [10, Corollary 1.4], $B(U) \cong H^1(C, \mathbb{Q}/\mathbb{Z})$. It follows that ${}_2B(U)$ is isomorphic to $H^1(C, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{(2\kappa-2)}$. Let L denote the quotient field of S . We view S , U and \tilde{T} as subrings of L , all birational to each other.

The Brauer group of R is generated by the symbol algebras $(x, f_i)_m$. To determine the image of $B(R) \rightarrow B(S)$, we first consider the symbol $\Lambda = (x, v)_m$. Assume that we have extended the scalars so that Λ is a central simple L -algebra. Since $B(\tilde{T}) = 0$, we use the exact sequence (5) to measure the ramification of Λ at the prime divisors of \tilde{T} . We need only consider primes in the divisor $(x) + (v)$ on $\text{Spec } \tilde{T}$. Any prime of \tilde{T} containing v must also contain w , so let $I = (v, w)$. The element $x \in \tilde{T}$ is irreducible and $J = (x)$ is a prime ideal. The ramification map a agrees with the tame symbol (6) at the divisors I and J .

To compute the ramification $a_I(\Lambda)$ at the prime I , we notice that w is a local parameter in the local ring \tilde{T}_I . The valuation of v is 2, the valuation of x is 0. The tame symbol becomes x^2 . The residue field at I is the quotient field of \tilde{T}/I , which we identify with $k(x)$. Therefore

$$(20) \quad a_I(\Lambda) = k(x)(x^{2/m}).$$

In \tilde{T}/I the element x belongs to only one prime, so the ramification map r is

$$(21) \quad r_p(a_I(\Lambda)) = \begin{cases} 2 & \text{if } p = (x) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (20) represents an element of $H^1(k(x), \mathbb{Z}/m)$ of order $m/\gcd(m, 2)$.

Modulo $J = (x)$, \tilde{T} is isomorphic to $k[v, w]/(w^2 - v(a_3 - b_3v)(a_4 - b_4v) \cdots (a_n - b_nv))$, the affine coordinate ring of the non-singular curve C . By C we also denote the corresponding divisor on $\text{Spec } \tilde{T}$, and by $K(C)$ the function field of C . In \mathcal{O}_C , x is a local parameter and has valuation 1. The valuation of v is 0. The tame symbol (6) becomes v^{-1} . Therefore

$$(22) \quad a_J(\Lambda) = K(C)(v^{-1/m}).$$

To compute $r(K(C)(v^{-1/m}))$, notice that the divisor of v on C is $2P_2$, where P_2 is the prime divisor defined by $v = w = 0$. Therefore,

$$(23) \quad r_p(a_J(\Lambda)) = \begin{cases} -2 & \text{if } p = P_2 \\ 0 & \text{otherwise.} \end{cases}$$

For all m , (22) represents an element of order m in $H^1(K(C), \mathbb{Z}/m)$. Notice that for $m = 2$, the quadratic extension $a_J(\Lambda)$ corresponds to the element of ${}_2\text{Pic}(C)$ generated by the prime divisor P_2 .

To prove (a), assume $m = 2$. As in the proof of Proposition 2.12, let $U = \tilde{T}[x^{-1}]$. By the computations above we see that Λ ramifies only along the curve C . Therefore, by sequence

(5) applied to U , we see that the image of Λ in $B(L)$ lands in the image of $B(U) \rightarrow B(L)$. By Section 4.2, the group ${}_2\text{Pic}(C)$ is generated by the primes (v, w) , $(a_3 - b_3v, w)$, \dots , $(a_n - b_nv, w)$. Consider the subgroup H of order 2^{n-2} in ${}_2B(R)$ generated by the symbols $(x, y)_2$, $(x, f_3)_2$, \dots , $(x, f_n)_2$. Iterating the previous argument for $\Lambda_3 = (x, f_3)_2$, \dots , $\Lambda_n = (x, f_n)_2$, shows that the image of H under $B(R) \rightarrow B(L)$ is equal to the image of ${}_2B(U) \rightarrow B(L)$.

Since S is isomorphic to the localization $\tilde{T}[x^{-1}, w^{-1}]$, we view $\text{Spec } S$ as an open in $\text{Spec } \tilde{T} = \mathbb{A}^1 \times C$. The closed complement $\text{Spec } \tilde{T} - \text{Spec } S$ is $C + L_2 + \dots + L_n$, each L_i being a copy of \mathbb{A}^1 . The intersection numbers are $L_i \cdot L_j = 0$, $L_i \cdot C = 1$. The formula in part (b) for $B(S)$ follows from [10, Corollary 1.6]. By (20) and (21), the composite

$$B(R) \rightarrow B(S) \rightarrow \bigoplus_{i=2}^n H^1(\mathcal{O}(L_i)[x^{-1}], \mathbb{Q}/\mathbb{Z})$$

is onto. By (22) and (23), the cokernel of $B(R) \rightarrow B(S)$ is isomorphic to $H^1(C, \mathbb{Q}/\mathbb{Z})$ modulo the subgroup of elements annihilated by 2, which is (c). \square

Proposition 2.14. *Assume we are in the context of Proposition 2.12. Then ${}_2\text{Cl}(T)$ is isomorphic to $(\mathbb{Z}/2)^{(n-1)}$, is generated by the divisor classes of the prime ideals (f_1, z) , \dots , (f_{n-1}, z) , and*

$$H^i(G, \text{Cl}(T)) \cong \begin{cases} \text{Cl}(T) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 & \text{if } i \text{ is odd} \\ \text{Cl}(T)^G = {}_2\text{Cl}(T) \cong (\mathbb{Z}/2)^{(n-1)} & \text{if } i \text{ is even.} \end{cases}$$

Proof. In T the only height one prime containing f_j is $I_j = (f_j, z)$. A local parameter for the discrete valuation ring T_{I_j} is z . One checks that the divisor of f_j is $2I_j$. Using (13), we see that $T[f_1^{-1}]$ is isomorphic to $k[x, x^{-1}] \otimes \mathcal{O}(C)$, where C is the affine hyperelliptic curve $w^2 = v(a_3 - b_3v) \cdots (a_n - b_nv)$. The group $\mathcal{O}^*(C)$ of units on C is equal to k^* , because for a nonsingular completion \bar{C} of C there is only one point at infinity $\bar{C} - C$. It follows that the group of units of $T[f_1^{-1}]$ is $k^* \times \langle f_1 \rangle$ and the class group of $T[f_1^{-1}]$ is isomorphic to the class group of C . The groups $H^i(G, \text{Cl}(T[f_1^{-1}]))$ are as in (15). The genus of C is $\kappa - 1$, where $\kappa = n/2$. By Section 4.2, the group ${}_2\text{Cl}(C)$, has order 2^{n-2} and the divisors (v, w) , $(a_3 - b_3v, w)$, \dots , $(a_{n-1} - b_{n-1}v, w)$ are a $\mathbb{Z}/2$ -basis for ${}_2\text{Cl}(C)$. Nagata's sequence [12, Theorem 7.1] becomes

$$(24) \quad 0 \rightarrow \mathbb{Z}/2 \xrightarrow{\alpha} \text{Cl}(T) \xrightarrow{\beta} \text{Cl}(T[f_1^{-1}]) \rightarrow 0$$

where α maps 1 to the divisor class generated by I_1 . Under β , we have the assignments of divisor classes $I_2 \mapsto (v, w)$, $I_3 \mapsto (a_3 - b_3v, w)$, \dots , $I_n \mapsto (a_n - b_nv, w)$. We see that the divisors I_1, \dots, I_n generate a subgroup of $\text{Cl}(T)$ of order 2^{n-1} . The diagram of abelian groups

$$(25) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\alpha} & \text{Cl}(T) & \xrightarrow{\beta} & \text{Cl}(T[f_1^{-1}]) \longrightarrow 0 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\alpha} & \text{Cl}(T) & \xrightarrow{\beta} & \text{Cl}(T[f_1^{-1}]) \longrightarrow 0 \end{array}$$

is commutative, where the vertical maps are “multiplication by 2”. The group $\text{Cl}(T[f_1^{-1}])$ is divisible. The Snake Lemma and the previous computations imply that ${}_2\text{Cl}(T) \cong$

$(\mathbb{Z}/2)^{(n-1)}$ and $\text{Cl}(T) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$. A $\mathbb{Z}/2$ -basis for ${}_2\text{Cl}(T)$ is the set of divisor classes of I_1, \dots, I_{n-1} . The long exact sequence of cohomology associated to (24) is

$$(26) \quad 0 \rightarrow \mathbb{Z}/2 \xrightarrow{\alpha} \text{Cl}(T)^G \xrightarrow{\beta} \text{Cl}(T[f_1^{-1}])^G \\ \xrightarrow{\partial^1} \mathbb{Z}/2 \xrightarrow{\alpha^1} H^1(G, \text{Cl}(T)) \xrightarrow{\beta^1} H^1(G, \text{Cl}(T[f_1^{-1}])) \\ \xrightarrow{\partial^2} \mathbb{Z}/2 \xrightarrow{\alpha^2} H^2(G, \text{Cl}(T)) \xrightarrow{\beta^2} H^2(G, \text{Cl}(T[f_1^{-1}])) \\ \xrightarrow{\partial^3} \mathbb{Z}/2 \xrightarrow{\alpha^3} H^3(G, \text{Cl}(T)) \xrightarrow{\beta^3} H^3(G, \text{Cl}(T[f_1^{-1}])) \xrightarrow{\partial^4} \dots$$

The terms with $T[f_1^{-1}]$ come from (15). Since $\beta : {}_2\text{Cl}(T) \rightarrow {}_2\text{Cl}(T[f_1^{-1}])$ is onto, we see that $\text{Cl}(T)^G = {}_2\text{Cl}(T)$ and α^1 is an isomorphism. Periodicity implies α^3 is an isomorphism. \square

Remark 2.15. Propositions 2.2 and 2.14 imply that $B(S/R) \cong H^1(G, \text{Cl}(T))$.

Proposition 2.16. *In the context of Proposition 2.12, let $X = \text{Spec } T$ and let P be the singular point of X . The sequence*

$$0 \rightarrow \mathbb{Z}/2 \rightarrow {}_2B(R) \rightarrow {}_2B(X-P) \rightarrow 0$$

is exact, where ${}_2B(X-P)$ is viewed as a subset of $B(S)$.

Proof. Let X_i be the open in X defined by $f_i \neq 0$. By Proposition 2.13(a),

$$0 \rightarrow \mathbb{Z}/2 \rightarrow {}_2B(R) \rightarrow {}_2B(X_i) \rightarrow 0$$

is exact for $i = 1, 2$. Therefore, as a subsets of $B(S)$, ${}_2B(X_1) = {}_2B(X_2)$. The ideal (f_1, f_2) is primary for the maximal ideal (x, y, z) . Therefore, $X_1 \cup X_2 = X - P$. The Mayer-Vietoris sequence

$$0 \rightarrow B(X_1 \cup X_2) \rightarrow B(X_1) \oplus B(X_2) \rightarrow B(X_1 \cap X_2)$$

is exact [22, Exercise III.2.24]. \square

3. AN AFFINE DOUBLE PLANE RAMIFIED ALONG A HYPERELLIPTIC CURVE

Let $f = y^2 - p(x)$, where $p(x) \in k[x]$ is monic, and has degree $d \geq 2$. Let $A = k[x, y]$, $T = A[z]/(z^2 - f)$, $R = A[f^{-1}]$, $S = T[z^{-1}]$. Let $p(x) = \ell_1^{e_1} \cdots \ell_v^{e_v}$ be the unique factorization of $p(x)$. Assume ℓ_1, \dots, ℓ_v are distinct and of the form $\ell_i = x - \alpha_i$. Let $D = \gcd(e_1, \dots, e_v)$. Let $F = Z(y^2 - p(x))$ be the subset of \mathbb{A}^2 . Then F is nonsingular if and only if each $e_i = 1$. The singular locus of the curve F is equal to the set of points $\{(\alpha_i, 0) \mid e_i \geq 2\}$. Let P_i be the point on $\text{Spec } T$ where $x = \alpha_i, y = z = 0$.

Proposition 3.1. *In the above context, the following are true.*

- (a) *The singular locus of T is equal to the set $\{P_i \mid e_i \geq 2\}$.*
- (b) *T is rational.*
- (c) *$\text{Cl}(T) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)}$.*
- (d) *$k^* = T^*$.*
- (e) *If $e_i \geq 2$, the singular point P_i of T is a rational double point of type A_n , where $n = e_i - 1$.*
- (f) *The natural homomorphism $\text{Cl}(T_{P_i}) \rightarrow \text{Cl}(\hat{T}_{P_i})$ is an isomorphism. Both groups are cyclic of order e_i .*

Proof. The singular locus of T consists of the set of points lying over the singular locus of F , from which (a) follows. We have $z^2 - f = z^2 - y^2 + p(x) = (z - y)(z + y) + p(x)$. Setting $u = z - y$ and $t = z + y$ defines an isomorphism of k -algebras

$$(27) \quad T = \frac{k[x, y, z]}{(z^2 - y^2 + p(x))} \xrightarrow{\alpha} \frac{k[x, u, t]}{(ut + p(x))}$$

where α sends $z \mapsto (u + t)/2$, $y \mapsto (t - u)/2$, $x \mapsto x$. Upon inverting u and eliminating t , (27) gives rise to the isomorphism

$$(28) \quad \frac{k[x, u, t][u^{-1}]}{(ut + p(x))} \xrightarrow{\beta} k[x, u, u^{-1}]$$

where β sends $t \mapsto -p(x)u^{-1}$, $u \mapsto u$, $x \mapsto x$. This shows T is rational, which is (b).

The ring on the right hand side of (28) is factorial, hence so is $T[(z - y)^{-1}]$. Nagata's Theorem [12, Theorem 7.1] says the class group of T is generated by the prime divisors that contain $z - y$. There are v minimal primes of $z - y$ in T , namely $(z - y, \ell_1), \dots, (z - y, \ell_v)$. Let $I_i = (z - y, \ell_i)$. In the discrete valuation ring T_{I_i} , $z - y = -(z + y)^{-1}p(x)$. Therefore ℓ_i generates the maximal ideal and the valuation of $z - y$ is e_i . The divisor of $z - y$ on $\text{Spec } T$ is $\text{div}(z - y) = e_1 I_1 + \dots + e_v I_v$. Using the isomorphism (28), we know that the group of units of $T[(z - y)^{-1}]$ is equal to $k^* \times \langle z - y \rangle$. Nagata's Theorem yields the exact sequence

$$(29) \quad 1 \rightarrow T^* \rightarrow T[(z - y)^{-1}]^* \xrightarrow{\text{div}} \bigoplus_{i=1}^v \mathbb{Z} I_i \rightarrow \text{Cl}(T) \rightarrow 0$$

If $D = \gcd(e_1, \dots, e_v)$, then $\text{Cl}(T) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)}$, which is (c). Part (d) follows from (29).

The natural map $\text{Cl}(T) \rightarrow \text{Cl}(T_{P_i})$ is onto, by Nagata's Theorem. For each $j \neq i$, ℓ_j is invertible in T_{P_i} . The group $\text{Cl}(T_{P_i})$ is cyclic and is generated by the divisor I_i . The isomorphism α in (27) shows that the complete local ring \hat{T}_{P_i} is isomorphic to the completion of $k[a, b, c]/(c^{n+1} + ab)$ at the maximal ideal (a, b, c) , where $n = e_i - 1$. The map sends $z - y \mapsto a$, $z + y \mapsto b$ and $\ell_i \mapsto c\gamma$, where γ is invertible. Part (e) follows from [6, A5]. The class group of $k[a, b, c]/(c^{n+1} + ab)$ is cyclic of order $e_i = n + 1$ and is generated by the ideal (c, a) . Therefore, the class group of \hat{T}_{P_i} is cyclic of order $e_i = n + 1$ and generated by the ideal I_i . This implies the composite homomorphism

$$(30) \quad \text{Cl}(T) \rightarrow \text{Cl}(T_{P_i}) \rightarrow \text{Cl}(\hat{T}_{P_i})$$

is onto. By Mori's Theorem [12, Corollary 6.12], the second map in (30) is one-to-one, so we get (f). \square

In Proposition 3.2, let $X = \text{Spec } T$ and $P = \{P_1, \dots, P_v\}$. By Proposition 3.1(a), the singular locus of X is a (possibly empty) subset of P . By a desingularization of P we mean a proper, surjective, birational morphism $\tau : \tilde{X} \rightarrow X$ such that \tilde{X} is nonsingular and τ is an isomorphism on $\tilde{X} - E$, where $E = \tau^{-1}(P)$.

Proposition 3.2. *In the above notation, with $X = \text{Spec } T$, the following are true for any desingularization $\tau : \tilde{X} \rightarrow X$.*

- (a) $0 = \text{B}(X) = \text{H}^2(X, \mathbb{G}_m)$.
- (b) $0 = \text{B}(\tilde{X}) = \text{H}^2(\tilde{X}, \mathbb{G}_m)$.
- (c) $0 = \text{B}(X - P)$.

	$(1, 0)$	$(z, 0)$	$(0, \ell_i)$	$(0, z - y)$
$(1, 0)$	$(1, 0)$	$(z, 0)$	$(0, f_i)$	$(0, z - y)$
$(z, 0)$	$(z, 0)$	$(y^2 - p(x), 0)$	$(0, z\ell_i)$	$(0, z(z - y))$
$(0, \ell_i)$	$(0, \ell_i)$	$(0, -z\ell_i)$	$(\ell_i, 0)$	$(-(z + y), 0)$
$(0, z - y)$	$(0, z - y)$	$(0, -z(z - y))$	$(z - y, 0)$	$(-p(x)\ell_i^{-1}, 0)$

TABLE 1. Multiplication table for $\Delta(I_i)$ in Remark 3.5.

Proof. All of the claims follow from results in [7]. Using (28) we see that the Brauer group of the ring $T[(z - y)^{-1}]$ is trivial. If L denotes the quotient field of T , the natural map $B(T) \rightarrow B(L)$ is one-to-one and factors through $B(T) \rightarrow B(T[(z - y)^{-1}])$. \square

Proposition 3.3. *In the above context, $\text{Pic } T$ is isomorphic to the subgroup of $\text{Cl}(T)$ generated by the ideal classes $e_1 I_1, \dots, e_v I_v$. It is a free \mathbb{Z} module of rank $v - 1$.*

Proof. By (30) we know that $\text{Cl}(T) \rightarrow \text{Cl}(\hat{T}_P)$ is onto. Also, the image of I_i in $\text{Cl}(\hat{T}_P)$ has order e_i if $i = j$, and order 1 otherwise. Therefore the sequence

$$(31) \quad 0 \rightarrow \text{Pic}(T) \rightarrow \text{Cl}(T) \rightarrow \bigoplus_{i=1}^v \text{Cl}(\hat{T}_P) \rightarrow 0$$

of [7] is exact. \square

Proposition 3.4. *In the above context, $H^1(G, \text{Cl}(T)) \cong B(S/R)$. For all $j \geq 0$,*

$$H^{2j+1}(G, \text{Cl}(T)) \cong \begin{cases} (\mathbb{Z}/2)^{(v)} & \text{if } D \equiv 0 \pmod{2}, \\ (\mathbb{Z}/2)^{(v-1)} & \text{else.} \end{cases}$$

and

$$H^{2j}(G, \text{Cl}(T)) \cong \text{Cl}(T)^G \cong \begin{cases} \mathbb{Z}/2 & \text{if } D \equiv 0 \pmod{2}, \\ 0 & \text{else.} \end{cases}$$

Proof. By Proposition 3.1, we are in the context of [11, Theorem 2.7]. Then $H^1(T, \text{Cl}(T))$ is isomorphic to $B(S/R)$. If σ is the generator for G , then σ sends every non-identity element of $\text{Cl}(T)$ to its inverse. This and [11, Theorem 2.4] give the rest. \square

Remark 3.5. In Proposition 3.4, the isomorphism $\Delta : H^1(T, \text{Cl}(T)) \cong B(S/R)$ is defined in [11, §2.2]. We sketch the construction in our present context. We restrict our description to one generator, the class of I_i in $\text{Cl}(T)$. Notice that $I_i \sigma(I_i) = (z^2 - y^2, (z - y)\ell_i, (z + y)\ell_i, \ell_i^2) \subseteq (\ell_i)$. As an A -module, $\Delta(I_i)$ is the direct sum $T \oplus I_i$. The multiplication rule on $\Delta(I_i)$ is defined on two arbitrary ordered pairs (a, b) and (c, d) by the formula

$$(32) \quad (a, b)(c, d) = (ac + b\sigma(d)\ell_i^{-1}, b\sigma(c) + ad).$$

This makes $\Delta(I_i)$ into an A -algebra. Upon localizing to S , the ideal I_i is projective. The R -algebra $\Delta(I_i) \otimes_A R$ is a generalized crossed product which is an Azumaya R -algebra [20]. It is not too hard to show that the ideal $I_i = (z - y, \ell_i)$ is generated as an A -module by $z - y$ and ℓ_i . From this it follows that I_i is a free A -module. Therefore, $\Delta(I_i)$ is a free A -module of rank 4. Is it true that every height one prime of T is a free A -module? Using equation (32), the multiplication table for $\Delta(I_i)$ is constructed in Table 1. Upon restricting to the quotient field K of A , it is clear that $\Delta(I_i) \otimes_A K$ is the symbol algebra $(y^2 - p(x), \ell_i)_2$.

Proposition 3.6. *In the above context, the following are true.*

$$(a) \operatorname{Pic}(S) \cong \begin{cases} \mathbb{Z}/(D/2) \oplus \mathbb{Z}^{(v-1)} & \text{if } D \equiv 0 \pmod{2}, \\ \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)} & \text{else.} \end{cases}$$

(b) For all $j \geq 0$,

$$H^{2j+1}(G, \operatorname{Pic}(S)) \cong \begin{cases} (\mathbb{Z}/2)^{(v)} & \text{if } D \equiv 0 \pmod{4}, \\ (\mathbb{Z}/2)^{(v-1)} & \text{else.} \end{cases}$$

and

$$H^{2j}(G, \operatorname{Pic}(S)) \cong \operatorname{Pic}(S)^G = {}_2\operatorname{Pic}(S) \cong \begin{cases} \mathbb{Z}/2 & \text{if } D \equiv 0 \pmod{4}, \\ 0 & \text{else.} \end{cases}$$

$$(c) S^* = \begin{cases} k^* \times \langle z \rangle \times \langle y - \sqrt{p(x)} \rangle & \text{if } D \equiv 0 \pmod{2}, \\ k^* \times \langle z \rangle & \text{else.} \end{cases}$$

(d) $H^0(G, S^*) = R^* = k^* \times \langle f \rangle$. For all $j > 0$,

$$H^{2j-1}(G, S^*) = \langle 1 \rangle$$

and

$$H^{2j}(G, S^*) \cong \begin{cases} \mathbb{Z}/2 & \text{if } D \equiv 0 \pmod{2}, \\ \langle 1 \rangle & \text{else.} \end{cases}$$

Proof. Using [11, Theorem 2.4], (b) follows from (a). Using [25, Theorem 10.35], (d) follows from (c). By the isomorphism (28), the image of z in the ring $k[x, u, u^{-1}]$ is $(u^2 - p(x))(2u)^{-1}$. Therefore,

$$(33) \quad S[(z-y)^{-1}] = T[z^{-1}, (z-y)^{-1}] \xrightarrow{\beta\alpha} k[x, u][u^{-1}, (u^2 - p(x))^{-1}]$$

is an isomorphism. There are two cases, which we treat separately.

Case 1: D is odd. Then the affine curve $y^2 - p(x)$ is irreducible and we are in the context of [11, Theorem 2.8]. In particular, $\operatorname{Cl}(T) = \operatorname{Pic}(S)$, and $S^* = k^* \times \langle z \rangle$, which gives the second halves of (a) of (c).

Case 2: D is even. Then we can factor $p(x) = q(x)^2$ and we get

$$(34) \quad R^* = k^* \times \langle y - q \rangle \times \langle y + q \rangle$$

The group of units in the ring on the right hand side of (33) is equal to $k^* \times \langle u \rangle \times \langle u + q \rangle \times \langle u - q \rangle$. Since $z - y \mapsto u$, $z - y + q \mapsto u + q$, and $z - y - q \mapsto u - q$, we have

$$(35) \quad S[(z-y)^{-1}]^* = k^* \times \langle z - y \rangle \times \langle z - y + q \rangle \times \langle z - y - q \rangle$$

The rings in (33) are factorial. We repeat the computation in (29), but with the rings S and $S[(z-y)^{-1}]$. First we determine the minimal primes of T containing $z - y$, $z - y + q$, and $z - y - q$. Notice that

$$(36) \quad (z - y + q)(z - y - q) = 2z(z - y)$$

$$(37) \quad (z - y)(z + y) = p(x) = \ell_1^{e_1} \cdots \ell_v^{e_v}$$

By (36), any prime that contains $z - y + q$ also contains z , or $z - y$. Let $P_1 = (z - y + q, z) = (z, y - q)$. One checks that P_1 is a height one prime of T and that in the local ring T_{P_1} , we have $v_{P_1}(z) = 1$, $v_{P_1}(y - q) = 2$, $v_{P_1}(z - y + q) = 1$. Any ideal that contains $z - y + q$ and $z - y$ also contains q . There are v minimal primes of $(z - y, q)$. They are $I_i = (z - y, \ell_i)$ for $i = 1, \dots, v$. By (37), ℓ_i generates the maximal ideal in the local ring T_{I_i} . Thus $v_{I_i}(\ell_i) = 1$, $v_{I_i}(z - y) = e_i$, $v_{I_i}(q) = e_i/2$. By (36), $e_i = v_{I_i}(z - y + q) + v_{I_i}(z - y - q)$. Use the fact that $v_{I_i}(z - y + q) \geq e_i/2$ and $v_{I_i}(z - y - q) \geq e_i/2$, to conclude that $v_{I_i}(z - y + q) = v_{I_i}(z - y -$

$q) = e_i/2$. The divisor of $z - y + q$ is $\text{div}(z - y + q) = P_1 + (e_1/2)I_1 + \cdots + (e_v/2)I_v$. Let $P_2 = (z, y + q)$. In the same way we get $\text{div}(z - y - q) = P_2 + (e_1/2)I_1 + \cdots + (e_v/2)I_v$ and $\text{div}(z - y) = P_1 + P_2$. Since P_1 and P_2 are not prime ideals in S , Nagata's sequence becomes

$$(38) \quad 1 \rightarrow S^* \rightarrow S[(z - y)^{-1}]^* \xrightarrow{\text{div}} \bigoplus_{i=1}^v \mathbb{Z}I_i \rightarrow \text{Pic}(S) \rightarrow 0$$

Evaluate div on the basis (35). The image of div is generated by the divisor $(e_1/2)I_1 + \cdots + (e_v/2)I_v$. This completes part (a). Using (38) we also find that S^* is generated over k^* by $(z - y + q)(z - y - q)(z - y)^{-1}$ and $(z - y + q)^2(z - y)^{-1}$. This and the identities (36) and (37) can be used to show that

$$(39) \quad S^* = k^* \times \langle z \rangle \times \langle y - q \rangle$$

which completes (c). \square

Proposition 3.7. *In the above context, the following are true.*

$$(a) \quad B(R) \cong \begin{cases} (\mathbb{Q}/\mathbb{Z})^{(v)} & \text{if } D \equiv 0 \pmod{2}, \\ (\mathbb{Q}/\mathbb{Z})^{(v-1)} & \text{else.} \end{cases}$$

(b) *In the exact sequence (4), the image of α_4 is*

$$B^\sim(S/R) \cong \begin{cases} \mathbb{Z}/2 & \text{if } D \equiv 2 \pmod{4}, \\ 0 & \text{else.} \end{cases}$$

(c) $B(S/R) = {}_2B(R)$.

(d) *The sequence*

$$0 \rightarrow B(S/R) \rightarrow B(R) \rightarrow B(S) \rightarrow 0$$

is exact. As a homomorphism of abstract groups, the map $B(R) \rightarrow B(S)$ is “multiplication by 2”.

Proof. In light of Proposition 3.4, to prove (c) it suffices to prove (a). We continue to use the notation established in the proof of Proposition 3.6. There are two cases, which we treat separately.

Case 1: D is odd. The map α_4 in (4) is the crossed product map. Since G is cyclic, crossed products are symbols of the form $(f, g)_2$, where $g \in R^*$. The image of α_4 is therefore generated by $(f, f)_2$, which is split. This is the second half of (b).

Rearrange the factors of $p(x) = \ell_1^{e_1} \cdots \ell_v^{e_v}$ so that e_i is odd for $1 \leq i \leq v_0$, and if $i > v_0$, then e_i is even. For each i , write $e_i = 2q_i + r_i$, where $0 \leq r_i < 2$. Let $r(x) = \ell_1^{r_1} \cdots \ell_{v_0}^{r_{v_0}}$. The normalization of $F = \text{Spec } k[x, y]/(y^2 - p(x))$ is the curve $\tilde{F} = \text{Spec } k[x, w]/(w^2 - r(x))$, where the map $\tilde{F} \rightarrow F$ is defined by $y \mapsto w\ell_1^{q_1} \cdots \ell_v^{q_v}$. Let P_i denote the closed point on F where $\ell_i = x - \alpha_i = 0$ and $y = 0$. Lying above P_i on \tilde{F} is the closed set defined by $x = \alpha_i$ and $w^2 = r(\alpha_i)$. For $1 \leq i \leq v_0$, there is only one point on \tilde{F} lying over P_i . For $i > v_0$, there are two points on \tilde{F} lying over P_i . By Section 4.2, $H^1(\tilde{F}, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{(v_0-1)}$. By [10, Corollary 1.6],

$$(40) \quad 0 \rightarrow H^1(\tilde{F}, \mathbb{Q}/\mathbb{Z}) \rightarrow H_F^3(\mathbb{A}^2, \mu) \rightarrow (\mathbb{Q}/\mathbb{Z})^{(v-v_0)} \rightarrow 0$$

is a split exact sequence. Therefore, $H_F^3(\mathbb{A}^2, \mu)$ is a free \mathbb{Q}/\mathbb{Z} -module of rank $v - 1$. By [10, Lemma 0.1], $B(R) \cong H_F^3(\mathbb{A}^2, \mu)$, which is the second half of (a).

The Brauer group of the ring on the right hand side of (33) is computed using [10, Corollary 3.2]. It is a free \mathbb{Q}/\mathbb{Z} -module of rank $2v - 1$. Since S is an affine surface, it follows from [10, Lemma 0.1] and [10, Corollary 1.6] that $B(S)$ is a free \mathbb{Q}/\mathbb{Z} -module of

finite rank. The reduced closed subset of $\text{Spec } S$ where $z - y = 0$ is the union of v copies of the one-dimensional torus. Therefore, $H^1(Z(z - y), \mathbb{Q}/\mathbb{Z})$ is a free \mathbb{Q}/\mathbb{Z} -module of rank v . By [10, Corollary 1.4], the sequence

$$(41) \quad 0 \rightarrow B(S) \rightarrow B(S[(z - y)^{-1}]) \rightarrow H^1(Z(z - y), \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

is exact. All of the groups in (41) are free \mathbb{Q}/\mathbb{Z} -modules of finite rank. Therefore, (41) splits and $B(S)$ is a free \mathbb{Q}/\mathbb{Z} -module of rank $v - 1$. This proves (d) in Case 1.

Case 2: D is even. Then $p = q^2$. Write $f_1 = y - q$ and $f_2 = y + q$. Let $F_1 = Z(f_1)$ and $F_2 = Z(f_2)$, both nonsingular rational curves in \mathbb{A}^2 . Let P_i be the point where $y = 0$ and $\ell_i = x - \alpha_i = 0$. The intersection $F_1 \cap F_2$ is equal to the set of v points P_1, \dots, P_v . Since $F_i \cong \mathbb{A}^1$, $H^1(F_i, \mathbb{Q}/\mathbb{Z}) = 0$. By [10, Corollary 1.4], $B(A[f_1^{-1}]) = 0$ and $B(A[f_2^{-1}]) = 0$. By [10, Corollary 3.2], $B(R) = B(A[f_1^{-1}, f_2^{-1}])$ is isomorphic to the direct sum of v copies of \mathbb{Q}/\mathbb{Z} , which is (a).

By (34), the image of α_4 in (4) is generated by the symbols $(f, f_1)_2 \sim (f_2, f_1)_2$ and $(f, f_2)_2 \sim (f_2, f_1)_2$, hence is cyclic. We use Theorem 1.1 to determine the exponent of $(f_1, f_2)_2$ in $B(R)$. Embed the curves in \mathbb{P}^2 , and let F_0 be the line at infinity. Let P_0 be the point at infinity where $y \neq 0$ and $x = 0$. Consider P_i , where $1 \leq i \leq v$. Because F_1 is nonsingular at P_i , we see that the local intersection multiplicity at P_i is $(F_1 \cdot F_2)_{P_i} = e_i/2$. Using this, we compute the weighted path in the graph $\Gamma(F_1 + F_2 + F_0)$ associated to the symbol $(f_1, f_2)_2$. Near the vertex P_i it looks like

$$F_1 \xrightarrow{e_i/2} P_i \xrightarrow{-e_i/2} F_2$$

If $4 \mid e_i$ for all i , then the symbol algebra $(f_1, f_2)_2$ is split. Otherwise, $(f_1, f_2)_2$ is an element of order two in the image of α_4 . This proves (b). Let $L_i = Z(\ell_i)$. Since F_1 is nonsingular at P_i , the intersection cycle on \mathbb{P}^2 is $F_1 \cdot L_i = P_i + (d/2 - 1)P_0$. Using this, the weighted path in the graph $\Gamma = \Gamma(F_1 + F_2 + F_0)$ associated to the symbol algebra $(f_1 f_2^{-1}, \ell_i)_m$ is

$$(42) \quad F_1 \rightarrow P_i \rightarrow F_2 \rightarrow P_0 \rightarrow F_1$$

For $i = 1, \dots, v$, the cycles (42) make up a basis for $H_1(\Gamma, \mathbb{Z}/m)$. Using Theorem 1.1, this proves that the algebras $(f_1 f_2^{-1}, \ell_i)_m$ form a basis for ${}_m B(R)$. Upon extending scalars to S , we have

$$(43) \quad (f_1 f_2^{-1}, \ell_i)_m \sim ((y^2 - p)/(y + q)^2, \ell_i)_m \sim (z/(y + q), \ell_i)_m^2$$

The image of $B(R) \rightarrow B(S)$ is divisible by two. To complete part (d), as in the previous case it is enough to show that $B(S)$ is a direct sum of v copies of \mathbb{Q}/\mathbb{Z} . We use sequence (41). As in the previous case, the group $H^1(Z(z - y), \mathbb{Q}/\mathbb{Z})$ is a free \mathbb{Q}/\mathbb{Z} -module of rank v . The ring in the right hand side of (33) is isomorphic to $R[y^{-1}]$. Use [10, Corollary 3.2] to see that $B(R[y^{-1}])$ is isomorphic to $B(R) \oplus (\mathbb{Q}/\mathbb{Z})^{(v)}$. Therefore, the rank of $B(S[(z - y)^{-1}])$ is equal to $2v$. By (41), $B(S)$ has rank v , completing the proof. \square

Remark 3.8. Consider the affine hyperelliptic curve $F = Z(y^2 - (x - \alpha_1)^{e_1} \cdots (x - \alpha_v)^{e_v})$. For the affine double plane $X \rightarrow \mathbb{A}^2$ that ramifies along F , and the invariants of X computed in Section 3, the roots $\alpha_1, \dots, \alpha_v$ do not seem to matter, whereas the number of roots v and multiplicities e_1, \dots, e_v of the roots play a role. What invariants of X depend on the actual roots $\alpha_1, \dots, \alpha_v$?

4. DIVISORS ON HYPERELLIPTIC CURVES

4.1. Divisors on a projective hyperelliptic curve. In this section Y denotes a nonsingular integral projective hyperelliptic curve and $\pi : Y \rightarrow \mathbb{P}^1$ is a double cover with ramification locus $Q = \{Q_1, \dots, Q_n\}$ on Y . Using the Riemann-Hurwitz Formula [16, Corollary IV.2.4], it follows that n is an even integer greater than or equal to 4 and the genus of Y is $g = (n-2)/2$. By Kummer Theory, the group $H^1(Y, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{(n-2)}$ classifies the cyclic Galois covers of Y and can be identified with the torsion subgroup of $\text{Pic}(Y)$ [22, Proposition III.4.11]. Let P_i denote the image $\pi(P_i)$ on \mathbb{P}^1 and $P = \{P_1, \dots, P_n\}$. Consider $U = \mathbb{P}^1 - P$ and $X = Y - Q$. Then $\pi : X \rightarrow U$ is a quadratic Galois cover. The affine coordinate ring $\mathcal{O}(U)$ is isomorphic to $k[x][f^{-1}]$, where f factors into $n-1$ distinct linear polynomials. So $\text{Pic}(U) = 0$. There is a basis for $U^*/k^* = H^0(U, \mathbb{G}_m)/k^*$ corresponding to the principal divisors $P_1 - P_n, \dots, P_{n-1} - P_n$ on \mathbb{P}^1 . Denote by f_i an element of $K(U)$ such that the divisor of f_i on \mathbb{P}^1 is $\text{div}(f_i) = P_i - P_n$. Then $U^*/k^* = \langle f_1 \rangle \times \dots \times \langle f_{n-1} \rangle$. By Kummer Theory,

$$(44) \quad U^*/(U^*)^2 \cong H^1(U, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{(n-1)}.$$

Consider the natural maps

$$(45) \quad \begin{array}{ccc} & H^1(U, \mathbb{Z}/2) & \\ & \downarrow \pi^* & \\ H^1(Y, \mathbb{Z}/2) & \xrightarrow{\beta} & H^1(X, \mathbb{Z}/2) \end{array}$$

By [8, Lemma 3.11], the images $\pi^*(f_i)$ generate a $\mathbb{Z}/2$ -submodule of $H^1(X, \mathbb{Z}/2)$ of rank $n-2$ and each image $\pi^*(f_i)$ is in the image of β . Therefore the image of β is generated by $\pi^*(f_1), \dots, \pi^*(f_{n-1})$. It follows that ${}_2\text{Pic}(Y)$ is generated by the divisors $Q_1 - Q_n, \dots, Q_{n-1} - Q_n$. The kernel of π^* is cyclic and corresponds to the quadratic Galois cover $\pi : X \rightarrow U$. On the function fields, this corresponds to adjoining the square root of $f_1 \cdots f_{n-1}$ to $K(\mathbb{P}^1)$. If $z^2 = f_1 \cdots f_{n-1}$, then $\text{div}(z) = Q_1 + Q_2 + \dots + Q_{n-1} - (n-1)Q_n$ is a principal divisor on Y . This shows that

$$(46) \quad \{Q_1 - Q_n, \dots, Q_{n-2} - Q_n\}$$

is a $\mathbb{Z}/2$ -basis for ${}_2\text{Pic}(Y)$, and the group of units on X is

$$(47) \quad \mathcal{O}^*(X) = k^* \times \langle z \rangle \times \langle f_2 \rangle \cdots \times \langle f_{n-1} \rangle.$$

4.2. Divisors on an affine hyperelliptic curve. In this section we consider an affine hyperelliptic curve. Let $n \geq 4$ be an integer. Let $\lambda_1, \dots, \lambda_n$ be distinct elements of k and set $f(x) = (x - \lambda_1) \cdots (x - \lambda_n)$. Let $X = Z(y^2 - f(x))$, an affine hyperelliptic curve in \mathbb{A}^2 . Let $\pi : X \rightarrow \mathbb{A}^1$ be the morphism induced by $k[x] \rightarrow \mathcal{O}(Y)$. Let Y be the complete nonsingular model for X and let $\pi : Y \rightarrow \mathbb{P}^1$ be the extension of π . Let Q_i denote the point on X (and on Y) where $y = x - \lambda_i = 0$. Let $P_i = \pi(Q_i)$. Employing the computations from Section 4.1, there are two cases.

Case 1: n is even. Then $Y - X$ consists of two points, say Q_{01} and Q_{02} , and $\pi : Y \rightarrow \mathbb{P}^1$ ramifies only on the set of n points $Q = \{Q_1, \dots, Q_n\}$. The genus of Y is $(n-2)/2$. By (46), the divisor classes $Q_1 - Q_n, \dots, Q_{n-2} - Q_n$ form a $\mathbb{Z}/2$ -basis for $H^1(Y, \mathbb{Z}/2) = {}_2\text{Pic}(Y)$. Because X is affine, $H^2(X, \mathbb{Z}/2) = 0$. Because Y is projective, $H^2(Y, \mathbb{Z}/2) =$

$\mathbb{Z}/2$. Because Y is nonsingular,

$$H_{Q_{01}+Q_{02}}^p(Y, \mathbb{Z}/2) = \begin{cases} 0 & \text{if } p < 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } p = 2 \end{cases}$$

by Cohomological Purity [22, Theorem VI.6.1]. The sequence of cohomology with supports in $Y - X$ is

$$(48) \quad 0 \rightarrow H^1(Y, \mathbb{Z}/2) \rightarrow H^1(X, \mathbb{Z}/2) \rightarrow H_{Q_{01}+Q_{02}}^2(Y, \mathbb{Z}/2) \rightarrow H^2(Y, \mathbb{Z}/2) \rightarrow 0$$

which implies

$$(49) \quad H^1(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{(n-1)}.$$

Let $P = \{P_1, \dots, P_n\}$, $V = Y - Q$, and $U = \mathbb{A}^1 - P$. The group of units on U is $U^* = k^* \times \langle x - \lambda_1 \rangle \times \dots \times \langle x - \lambda_n \rangle$. By Kummer Theory, $H^1(U, \mathbb{Z}/2)$ is isomorphic to U^*/U^{*2} . The counterpart of diagram (45) for the Galois cover of affine curves $\pi : V \rightarrow U$ is

$$(50) \quad \begin{array}{ccc} & H^1(U, \mathbb{Z}/2) & \\ & \downarrow \pi^* & \\ H^1(X, \mathbb{Z}/2) & \xrightarrow{\beta} & H^1(V, \mathbb{Z}/2) \end{array}$$

The image of π^* has $\mathbb{Z}/2$ -rank $n - 1$ and is contained in the image of β , which also has $\mathbb{Z}/2$ -rank $n - 1$. This proves that the image of π^* is equal to the image of β . This proves that the quadratic Galois extensions of X are all of the form

$$(51) \quad \mathcal{O}(X) \otimes_{k[x]} k[x]/(y^2 = (x - \lambda_1)^{e_1} \dots (x - \lambda_n)^{e_n})$$

where each e_i is 0 or 1. The quadratic extension in (51) is split if and only if $e_1 = e_2 = \dots = e_n$.

Case 2: n is odd. Then $Y - X = Q_0$ is a single point, and π ramifies at the points Q_0, Q_1, \dots, Q_n . The genus of Y is $(n - 1)/2$. The divisor classes $Q_1 - Q_0, \dots, Q_{n-1} - Q_0$ form a $\mathbb{Z}/2$ -basis for ${}_2\text{Pic}(Y)$. The Gysin sequence [22, Remark VI.5.4]

$$(52) \quad 0 \rightarrow H^1(Y, \mathbb{Z}/2) \rightarrow H^1(X, \mathbb{Z}/2) \rightarrow H_{Q_0}^2(Y, \mathbb{Z}/2) \rightarrow H^2(Y, \mathbb{Z}/2) \rightarrow 0$$

shows that

$$(53) \quad H^1(Y, \mathbb{Z}/2) \cong H^1(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{(n-1)}.$$

Therefore, Q_1, \dots, Q_{n-1} form a $\mathbb{Z}/2$ -basis for $H^1(X, \mathbb{Z}/2) = {}_2\text{Pic}(X)$.

Proposition 4.1. *As above, let $n \geq 3$ and X the affine hyperelliptic curve defined by $y^2 = \prod_{i=1}^n (x - \lambda_i)$. Let $\sigma : X \rightarrow X$ be the automorphism defined by $y \mapsto -y$. Let $G = \langle \sigma \rangle$. Then for all $i \geq 0$,*

- (a) $H^{2i+1}(G, \text{Pic}X) = 0$ and
- (b) $H^{2i}(G, \text{Pic}X) = (\text{Pic}X)^G = {}_2\text{Pic}X \cong (\mathbb{Z}/2)^{(r)}$. If n is odd, then r is equal to $n - 1$. If n is even, then r is either $n - 1$ or $n - 2$.

Proof. Every prime divisor on X is mapped by σ to its inverse. Since X is not rational, the only divisors that are fixed by G are the elements of order two in $\text{Pic}X$. That is, $(\text{Pic}X)^G = {}_2\text{Pic}X$. Following the notation of [25, pp. 296–297], let $N = \sigma + 1$ and $D = \sigma - 1$. It follows that $N : \text{Pic}X \rightarrow \text{Pic}X$ is the zero map and $D : \text{Pic}X \rightarrow \text{Pic}X$ is

multiplication by -2 . If ${}_N \text{Pic } X$ denotes the kernel of N , then by [25, Theorem 10.35] we have

$$H^{2i}(G, \text{Pic } X) = \frac{(\text{Pic } X)^G}{N \text{Pic } X} = (\text{Pic } X)^G,$$

the first part of (b), and

$$H^{2i+1}(G, \text{Pic } X) = \frac{{}_N \text{Pic } X}{D \text{Pic } X} = \text{Pic } X \otimes \mathbb{Z}/2.$$

By Kummer Theory, $\text{Pic } X \otimes \mathbb{Z}/2 \rightarrow H^2(X, \mu_2)$ is one-to-one. Since X is an affine curve, $H^2(X, \mu_2) = 0$, which proves (a). The rest of (b) follows from [8, Lemmas 3.1, 3.2]. \square

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